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DEVELOPMENT OF FREE CONVECTION FLOW OF A GAS IN A HEATED VERTICAL OPEN TUBE

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Abstract—The system is a vertical tube open at both ends and heated at the wall. An ambient gas ($Pr = 0{\cdot}7$) enters the bottom of the tube with uniform velocity and temperature and flows up through the tube due to natural convection. The flow is assumed to be both stable and laminar. The incompressible thermal boundary layer equations for this situation were solved by a finite difference method for conditions of constant wall temperature and constant wall heat flux.

From the velocity and temperature profiles obtained for various stages of the flow development, a graphical correlation was found between dimensionless tube length and two dimensionless quantities representative of the volumetric flow rate and the rate of heat dissipation. For the case of constant wall temperature these results were compared with those of Elenbaas [3], and excellent agreement was obtained. The results for constant wall heat flux were compared with those of Kays [8] on laminar forced convection. His calculations, in which the transverse velocity component was assumed negligible, gave Nusselt numbers near the entrance that are 20–30 per cent higher than the present results.

NOMENCLATURE

A, parameter in equation (25) equal to $2/F$;

A_k, $-\dfrac{V_{j,k}}{2\Delta R^2} - \dfrac{1}{(\Delta R)^2} + \dfrac{1}{2R_k \Delta R}$;

$\bar{A}_k$, $-\dfrac{V_{j,k}}{2\Delta R} - \dfrac{1}{Pr(\Delta R)^2} + \dfrac{1}{2Pr R_k \Delta R}$;

b, constant in equation (26);

B_k, $\dfrac{U_{j,k}}{\Delta Z} + \dfrac{2}{(\Delta R)^2}$;

$\bar{B}_k$, $\dfrac{U_{j,k}}{\Delta Z} + \dfrac{2}{Pr(\Delta R)^2}$;

ber (x), real part of $I_0(x\sqrt{i})$ = real part of $J_0(x\sqrt{i})$;

bei (x), imaginary part of $I_0(x\sqrt{i})$ = – imaginary part of $J_0(x\sqrt{i})$;

ber′ (x), d ber (x)/dx;

bei′ (x), d bei (x)/dx;

c, constant in equation (26);

c_i, imaginary part of c;

c_r, real part of c;

C_p, heat capacity of fluid;

C_k, $\dfrac{V_{j,k}}{2\Delta R} - \dfrac{1}{(\Delta R)^2} - \dfrac{1}{2R_k \Delta R}$;

$\bar{C}_k$, $\dfrac{V_{j,k}}{2\Delta R} - \dfrac{1}{Pr(\Delta R)^2} - \dfrac{1}{2Pr R_k \Delta R}$;

D, tube diameter;

D_k, $(U_{j,k}^2 + P_j)/\Delta Z$;

$\bar{D}_k$, $\dfrac{U_{j,k}}{\Delta Z} T_{j,k}$;

e, constant in equation (26);

e_i, imaginary part of e;

e_r, real part of e;

f, volumetric flow rate;

F, dimensionless volumetric flow rate;

g, body force in momentum equation (gravity);

Gr^*, modified Grashof number defined by equations (8) and (19);

$I_0(x)$, modified Bessel function of the first kind of order zero;

$J_0(x)$, Bessel function of the first kind of order zero;

k, thermal conductivity of fluid;

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$K_0(x)$, modified Bessel function of the second kind of order zero;
l, tube length;
L, dimensionless tube length $= 1/Gr^*$;
m, constant in equation (26);
n, counting variable;
Nu, Nusselt number;
p, pressure;
p_0, hydrostatic pressure of fluid at ambient conditions;
p', pressure defect as defined by equation (4);
P, dimensionless pressure defect;
Pr, Prandtl number;
q, heat dissipation rate;
$(q/A)_w$, heat flux at the wall;
Q, dimensionless heat dissipation rate;
Q', dimensionless heat dissipation rate evaluated at tube exit;
r, radial coordinate;
r_ω, radius of tube;
R, dimensionless radial coordinate;
R_k, dimensionless radial finite difference grid coordinate;
s, constant in equation (28) equal to $c_r + e_r$;
t, temperature at any point in the tube;
t_m, mixing cup temperature;
t_0, ambient temperature;
t_1, temperature of wall;
T, dimensionless temperature at any point in the tube;
T_{cl}, dimensionless temperature at center line of tube;
T_m, dimensionless mixing-cup temperature;
T'_m, dimensionless mixing-cup temperature evaluated at tube exit;
$T_{j,k}$, dimensionless temperature at grid coordinate (j, k);
$\bar{T}_0$, absolute ambient temperature;
u, fluid velocity in the flow direction;
u_0, fluid velocity in the flow direction at tube entrance;
U, dimensionless velocity in the flow direction;
v, fluid velocity in the radial direction;
V, dimensionless velocity in the radial direction
$V_{j,k}$, dimensionless velocity in the radial direction at grid coordinate (j, k);
w, constant in equation (27) equal to $c_i + e_i$;
$Y_0(x)$, Bessel function of the second kind of order zero;
z, flow direction coordinate;
Z, dimensionless flow direction coordinate.

Greek letters
α, thermal diffusivity;
ν, kinematic fluid viscosity;
ρ, fluid density;
ϕ, general term, representing U or T when introducing form of finite difference equations in equations (35)–(37).

INTRODUCTION

THE HANDLING of spent nuclear reactor fuel assemblies, which are often various tubular configurations and which generate heat by fission product decay, give rise to problems similar to the one considered here. Much of the previously published work has been concerned with either the effects of free convection on already developed laminar forced convection for confined flows [1, 2] or purely free convection for unconfined flow situations such as at a single vertical plate and around vertical and horizontal cylinders. Notable exceptions to the former are the investigations of Elenbaas [3] and Bodoia and Osterle [4]. Elenbaas carried out rather extensive analytical and experimental work on natural convective flow in such cross sectional geometries as the equilateral triangle, square, rectangle, circle and infinite parallel plates. The work of Bodoia and Osterle points up the need for additional investigation, especially for predeveloped flow. Their finite difference calculations on the development of free convective flow between heated vertical plates show that the development height is rather significant and that for most situations the assumption of fully developed flow is not valid. The present investigations extends the work of Bodoia and Osterle to natural convection in a vertical tube open at both ends with constant wall temperature. In addition the condition of constant wall heat flux was investigated.

Calculations were made for the velocity and temperature distributions throughout the tube assuming the fluid to enter at ambient temperature and with a flat velocity profile. The velocity and temperature distributions were obtained by solving the thermal boundary-layer equations in dimensionless form by a finite difference technique.

EQUATIONS DESCRIBING THE PROBLEM

Applying the usual boundary layer assumption [5] to the governing differential equations (continuity, momentum and energy) yields the following incompressible, two-dimensional, thermal boundary layer equations:

$$\frac{v}{r} + \frac{\partial v}{\partial r} + \frac{\partial u}{\partial z} = 0, \tag{1}$$

$$v\frac{\partial u}{\partial r} + u\frac{\partial u}{\partial z} = \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) - \frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}z} - g, \tag{2}$$

and

$$v\frac{\partial t}{\partial r} + u\frac{\partial t}{\partial z} = \alpha\left(\frac{\partial^2 t}{\partial r^2} + \frac{1}{r}\frac{\partial t}{\partial r}\right). \tag{3}$$

To make the above equations more amenable to solution for natural convective flow the body force in equation (2) can be expressed in terms of a bouyancy force. Also, density is assumed to vary only in the gravity force term and viscous dissipation is assumed negligible in the energy equation.

A pressure defect is defined as

$$p' = p - p_0 \tag{4}$$

where p_0 is the pressure that would result if the temperature were the same as ambient throughout the flow field. Thus, one can write

$$\frac{\mathrm{d}p_0}{\mathrm{d}z} = -\rho_0 g \tag{5}$$

which, when combined with equations (2) and (4) and assuming ideal gas behavior gives for the momentum equation

$$v\frac{\partial u}{\partial r} + u\frac{\partial u}{\partial z} = \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) - \frac{1}{\rho}\frac{\mathrm{d}p'}{\mathrm{d}z} + \frac{g(t - t_0)}{\overline{T}_0} \tag{6}$$

where $\overline{T}_0$ is the absolute ambient temperature.

Equations (1), (3) and (6) represent a considerable simplification of the original governing differential equations, yet still can not be solved analytically for pre-developed flow. For this reason a numerical solution was planned.

Constant wall temperature

To facilitate the numerical solution equations (1), (3) and (6) are written in dimensionless forms, by making the following substitutions:

$$\left.\begin{aligned} V &= \frac{r_\omega v}{\nu} \\ U &= \frac{r_\omega^2 u}{l\nu\, Gr^*}, \\ Z &= \frac{z}{l\, Gr^*} \\ R &= \frac{r}{r_\omega} \\ P &= \frac{p' r_\omega^4}{\rho l^2 \nu^2\, Gr^{*2}} \\ T &= \frac{t - t_0}{t_1 - t_0} \end{aligned}\right\} \tag{7}$$

where

$$Gr^* = \frac{g(t_1 - t_0) r_\omega^4}{\overline{T}_0 l \nu^2}. \tag{8}$$

The resulting dimensionless equations are continuity:

$$\frac{V}{R} + \frac{\partial V}{\partial R} + \frac{\partial U}{\partial Z} = 0 \tag{9}$$

momentum:

$$V\frac{\partial U}{\partial R} + U\frac{\partial U}{\partial Z} = \frac{\partial^2 U}{\partial R^2} + \frac{1}{R}\frac{\partial U}{\partial R} - \frac{\mathrm{d}P}{\mathrm{d}Z} + T, \tag{10}$$

and energy:

$$V\frac{\partial T}{\partial R} + U\frac{\partial T}{\partial Z} = \frac{1}{Pr}\left(\frac{\partial^2 T}{\partial R^2} + \frac{1}{R}\frac{\partial T}{\partial R}\right). \tag{11}$$

If an initial velocity (i.e. the velocity at $Z = 0$) is termed u_0 (constant, independent of R) then the volumetric flow rate can be written as

$$f = \pi r_\omega^2 u_0 = \int_0^{r_\omega} 2\pi r u \,\mathrm{d}r. \tag{12}$$

When the dimensionless variables in equations (7) are substituted into equation (12), the following dimensionless equation for the volumetric flow rate results:

$$F = \frac{f}{\pi l v\, Gr^*} = 2\int_0^1 UR\,\mathrm{d}R. \tag{13}$$

In a similar way the heat absorbed by the fluid rising in the tube can be written in dimensionless form as

$$Q = \frac{q}{\pi\rho C_p l v\, Gr^*(t_1 - t_0)} = 2\int_0^1 UTR\,\mathrm{d}R. \tag{14}$$

The boundary conditions for equations (9)–(11) can be written as

for $Z = 0$ and $0 \leqslant R < 1$:

$$U = F, \quad V = 0, \quad T = 0;$$

for $R = 0$ and $Z \geqslant 0$:

$$\frac{\partial U}{\partial R} = 0, \quad V = 0, \quad \frac{\partial T}{\partial R} = 0; \tag{15}$$

for $R = 1$ and $Z \geqslant 0$:

$$U = 0, \quad V = 0, \quad T = 1;$$

for $Z = 0$ and $Z = L$: $\quad P = 0.$

The limiting case of fully developed flow, which occurs when uniform temperature is achieved, provides an analytical check on the finite difference solution to be obtained. Noting that for developed flow, the dimensionless temperature $T = 1$, $\partial U/\partial Z = 0$, and $V = 0$, the momentum equation reduces to

$$\frac{\partial^2 U}{\partial R^2} + \frac{1}{R}\frac{\partial U}{\partial R} = -1 \tag{16}$$

which when solved for U, yields

$$U = \frac{1 - R^2}{4} \tag{17}$$

the characteristic parabolic velocity profile for isothermal Poiseuille flow. Hence, the limiting value of the dimensionless volumetric flow rate is

$$F = 2\int_0^1 \frac{1 - R^2}{4} R\,\mathrm{d}R = \tfrac{1}{8}. \tag{18}$$

Since $T = 1$, Q is similarly determined to be $\frac{1}{8}$.

Constant wall heat flux

For this condition it is necessary to redefine several of the dimensionless variables as follows: Let

$$T = \frac{(t - t_0)k}{(q/A)_\omega r_\omega}$$
$$Gr^* = \frac{g(q/A)_\omega r_\omega^5}{\bar{T}_0 l v^2 k} \tag{19}$$

and

$$Q = \frac{qk}{\pi\rho C_p l v\, Gr^*(q/A)_\omega r_\omega} = \frac{2}{Pr\, Gr^*}\frac{z}{l}.$$

The expressions for V, U, Z, R, P and F remain the same. The dimensionless equations resulting from substituting equations (19) and (7) into equations (1), (3) and (6) are identical to equations (9)–(11). The accompanying boundary conditions are for $Z = 0$ and $0 \leqslant R < 1$:

$$U = F, \quad V = 0, \quad T = 0;$$

for $R = 0$ and $Z \geqslant 0$:

$$\partial U/\partial R = 0, \quad V = 0, \quad \partial T/\partial R = 0; \tag{20}$$

for $R = 1$ and $Z \geqslant 0$:

$$U = 0, \quad V = 0, \quad \partial T/\partial R = 1;$$

for $Z = 0$ and $Z = L$: $\quad P = 0.$

For flow to be hydrodynamically and thermally developed, it is required that the velocity profile be invariant in the flow direction and that T vary linearly with Z. Under these conditions the momentum equation becomes

$$T = -\nabla^2 U + \frac{dP}{dZ}, \tag{21}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{1}{\partial R}.$$

Turning now to the energy equation, which can be written as

$$\frac{\partial}{\partial R}\left(R\frac{\partial T}{\partial R}\right) = RU\,Pr\frac{\partial T}{\partial Z}, \tag{22}$$

and integrating from the centerline to the wall

$$\frac{dT}{dZ} = \frac{2}{F\,Pr}. \tag{23}$$

Thus, equation (22) becomes

$$\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial T}{\partial R}\right) = \frac{2U}{F}. \tag{24}$$

Substituting equation (21) into equation (24) gives

$$\nabla^4 U + AU = 0 \tag{25}$$

where $A = 2/F$.

The solution to this equation can be represented by four Bessel functions as [2]

$$U = cJ_0(A^{\frac{1}{4}}R\sqrt{i}) + bY_0(A^{\frac{1}{4}}R\sqrt{i}) + eI_0(A^{\frac{1}{4}}R\sqrt{i}) + mK_0(A^{\frac{1}{4}}R\sqrt{i}). \tag{26}$$

Since the velocity is finite at $R = 0$,

$$b = 0 \text{ and } m = 0.$$

The functions $J_0(A^{\frac{1}{4}}R\sqrt{i})$ and $I_0(A^{\frac{1}{4}}R\sqrt{i})$ are complex numbers, where $J_0(AR\sqrt{i})$ is the complex conjugate of $I_0(AR\sqrt{i})$. Thus, one can write

$$U = (c_r + ic_i)\,J_0(A^{\frac{1}{4}}R\sqrt{i}) + (e_r + ie_i) \times I_0(A^{\frac{1}{4}}R\sqrt{i}). \tag{27}$$

The real and imaginary parts of the above two functions have been tabulated as the "ber" and "bei" functions [6], which allows equation (27) to be written as

$$U = s\,\mathrm{ber}\,(A^{\frac{1}{4}}R) + w\,\mathrm{bei}\,(A^{\frac{1}{4}}R) \tag{28}$$

where $s = (c_r + e_r)$ and $w = (c_i - e_i)$.

The constants s and w can be evaluated by using the conditions

$$U = 0 \quad \text{at } R = 1$$

and

$$F = 2\int_0^1 UR\,dR.$$

After considerable manipulation it was found that

$$s = \frac{A^{-\frac{1}{4}}\,\mathrm{bei}\,(A^{\frac{1}{4}})}{\mathrm{ber}'\,(A^{\frac{1}{4}})\,\mathrm{ber}\,(A^{\frac{1}{4}}) + \mathrm{bei}'\,(A^{\frac{1}{4}})\,\mathrm{bei}\,(A^{\frac{1}{4}})} \tag{29}$$

and

$$w = -s\frac{\mathrm{ber}\,(A^{\frac{1}{4}})}{\mathrm{bei}\,(A^{\frac{1}{4}})}. \tag{30}$$

Having the expression for velocity, equation (22) can be solved for T to give

$$T - T_{cl} = A^{\frac{1}{4}}[s\,\mathrm{bei}\,(A^{\frac{1}{4}}R) - w\,\mathrm{ber}\,(A^{\frac{1}{4}}R) + w]. \tag{31}$$

The evaluation of T cannot be carried any further since there is no *a priori* knowledge of T_{cl}. However, for purposes of comparing these results with those obtained from the numerical method this is sufficient, since T_{cl} is available there.

FINITE DIFFERENCE APPROXIMATION TO THE NATURAL CONVECTION EQUATIONS

The finite difference analysis of the dimensionless partial differential equations is begun by imposing a rectangular grid over the region to be investigated. The independent variables are then defined at the intersection of the grid lines, where (j, k) is a typical mesh point. Mesh points are numbered consecutively from an arbitrary origin with the j progressing in the flow (Z) direction and the k in the radial (R) direction. The dependent variables are designated as point functions by unique subscript pairs (j, k). Hence, the quantity $U(Z, R)$ is replaced by $U_{j,k}$.

For the partial derivatives appearing in the continuity equation, let

$$\frac{\partial V}{\partial R} = \frac{V_{j+1,k+1} - V_{j+1,k}}{\Delta R} \tag{32}$$

and

$$\frac{\partial U}{\partial Z} = \frac{U_{j+1,k+1} + U_{j+1,k} - U_{j,k+1} - U_{j,k}}{2\Delta Z}. \tag{33}$$

The symmetrical form used in equation (33) gives a smaller truncation error than unsymmetrical forms [7]. Substituting the above approximations into equation (9) yields:

$$\frac{V_{j,k}}{R_k} + \frac{V_{j+1,k+1} - V_{j+1,k}}{\Delta R} + \frac{U_{j+1,k} + U_{j+1,k+1} - U_{j,k+1} - U_{j,k}}{2\Delta Z} = 0. \tag{34}$$

For the momentum and energy equations the following finite difference approximations are used:

$$\frac{\partial \phi}{\partial R} = \frac{\phi_{j+1,k+1} - \phi_{j+1,k-1}}{2\Delta R}, \tag{35}$$

$$\frac{\partial \phi}{\partial Z} = \frac{\phi_{j+1,k} - \phi_{j,k}}{\Delta Z}, \tag{36}$$

and

$$\frac{\partial^2 \phi}{\partial R^2} = \frac{\phi_{j+1,k+1} - 2\phi_{j+1,k} + \phi_{j+1,k-1}}{(\Delta R)^2}; \tag{37}$$

where ϕ represents either U or T.

Substituting the above approximations into equations (10) and (11) yields:

$$A_k U_{j+1,k-1} + B_k U_{j+1,k} + C_k U_{j+1,k+1} + (P_{j+1})/\Delta Z = D_k + T_{j+1,k} \tag{38}$$

and

$$\bar{A}_k T_{j+1,k-1} + \bar{B}_k T_{j+1,k} + \bar{C}_k T_{j+1,k+1} = \bar{D}_k, \tag{39}$$

respectively. (See Nomenclature for A_k, B_k, C_k, D_k and $\bar{A}_k$, $\bar{B}_k$, $\bar{C}_k$, $\bar{D}_k$.)

Eliminating the $V_{j+1,k}$'s from equation (34) by means of successive substitutions, one has

$$\sum_{k=1}^{n} \frac{V_{j,k}}{R_k} + \frac{1}{2\Delta Z}\left[U_{j+1,0} - U_{j,0} + 2\sum_{k=1}^{n} (U_{j+1,k} - U_{j,k})\right] = 0 \tag{40}$$

where n represents the number of increments taken across the half tube (only half the tube need be considered since there is symmetry about the Z-axis). Equation (39) written for $j = 0$ and $k = 0, 1, \ldots, n-1$ becomes n equations with n unknowns which can be solved for $T_{j+1,0}, \ldots, T_{j+1,n-1}$. Knowing $T_{j+1,k}$, equations (38) and (40) which represent $n+1$ equations with $n+1$ unknowns, can now be solved for $U_{j+1,0}$, $U_{j+1,1}, \ldots, U_{j+1,n-1}$, and P_{j+1}. At this point equation (34) is used to determine the values of $V_{j+1,k+1}$. Having the velocity and temperature profiles at $Z = 1 \cdot \Delta Z$ one is in a position to repeat the calculations for the next level in the tube and so on up the tube until the dimensionless pressure returns to zero. At this point $Z = L$ and from the definition of Z this establishes $Gr^*(= 1/L)$.

Knowing the values of $U_{j,k}$ and $T_{j,k}$ permits the numerical integration of equations (13) and (14) to determine F and Q at each level of the tube.

RESULTS

The finite difference equations were solved for the velocity and temperature profiles at various

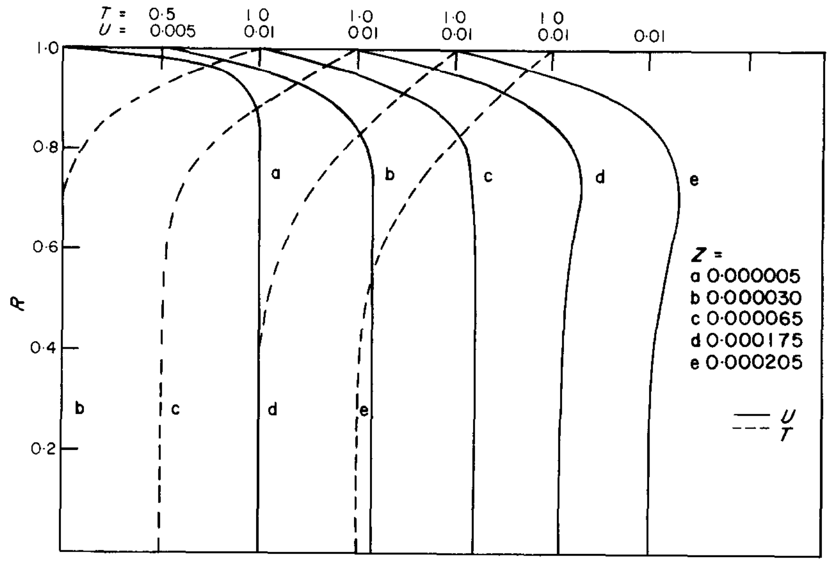

FIG. 1. Velocity and temperature profiles for $Q = 0{\cdot}00955$. Constant wall temperature.

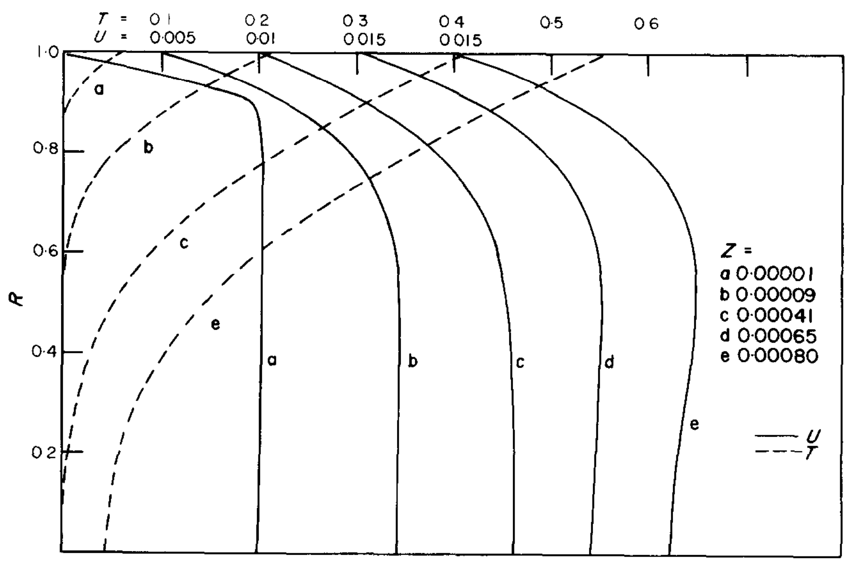

FIG. 2. Velocity and temperature profiles for $Q = 0{\cdot}0096$. Constant wall heat flux.

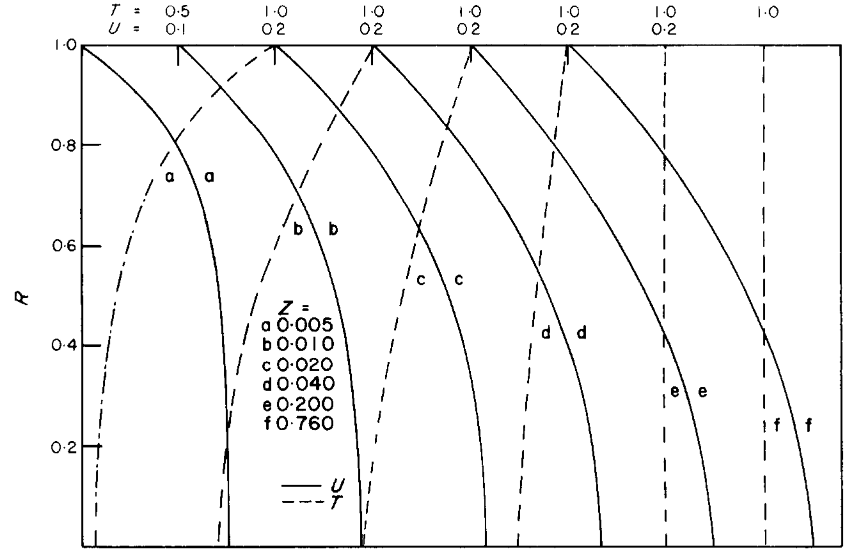

FIG. 3. Velocity and temperature profiles for $Q = 0{\cdot}12$. Constant wall temperature.

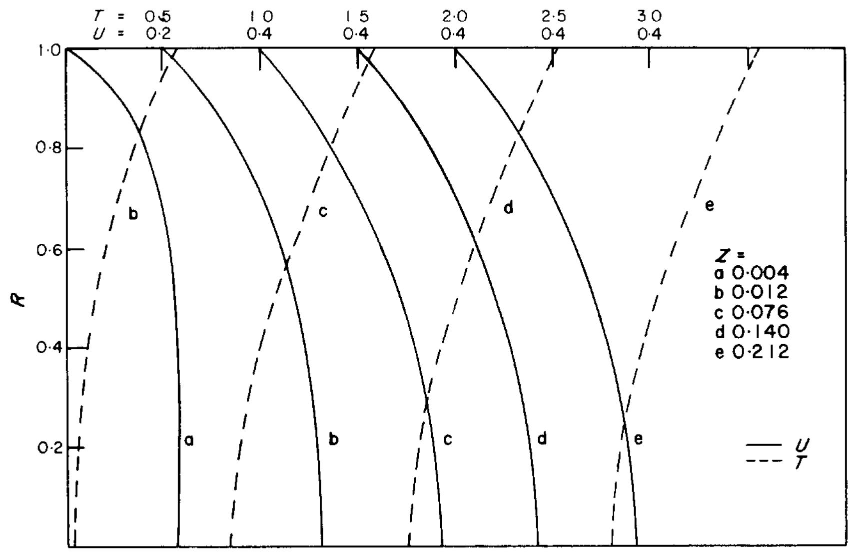

FIG. 4. Velocity and temperature profiles for $Q = 0{\cdot}193$. Constant heat wall flux.

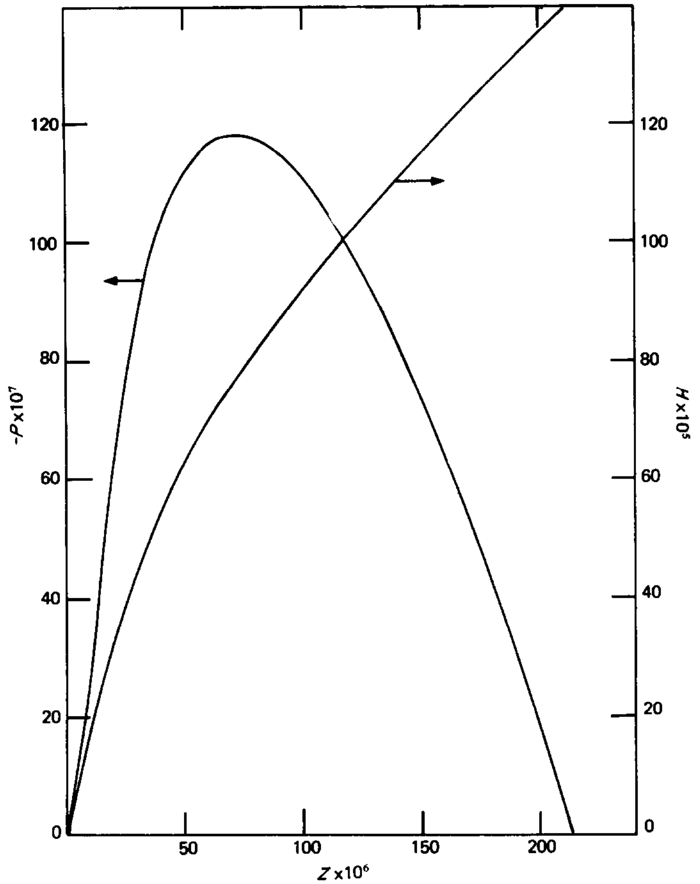

FIG. 5. Pressure defect and heat absorbed for $Q = 0{\cdot}00955$. Constant wall temperature.

stages of the flow development. Representative profiles are presented in Figs. 1–4. Figures 1 and 2 show the side developing temperature and velocity profiles associated with large diameter and high wall temperatures for constant wall temperature and constant wall heat flux, respectively. The curves in Figs. 3 and 4 indicate the centerline development resulting from relatively long, small diameter tubes for the two cases investigated.

Figures 5 and 6 show representative dimensionless heat flux and pressure levels as a function of axial position for constant wall temperature and constant wall heat flux, respectively. In either case, the point at which the pressure defect returns to zero defines the dimensionless tube length which in turn establishes the modified Grashof number. As one would expect the term Q increased at a decreasing rate for the constant wall temperature case and is linear for the constant wall heat flux case. For the majority of the calculations, for both constant wall temperature and constant wall heat flux, the Prandtl number was set equal to 0·7. In the case of constant wall temperature the range of F investigated was 0·0095–0·1238. For constant wall heat flux the range of F was 0·010–0·289.

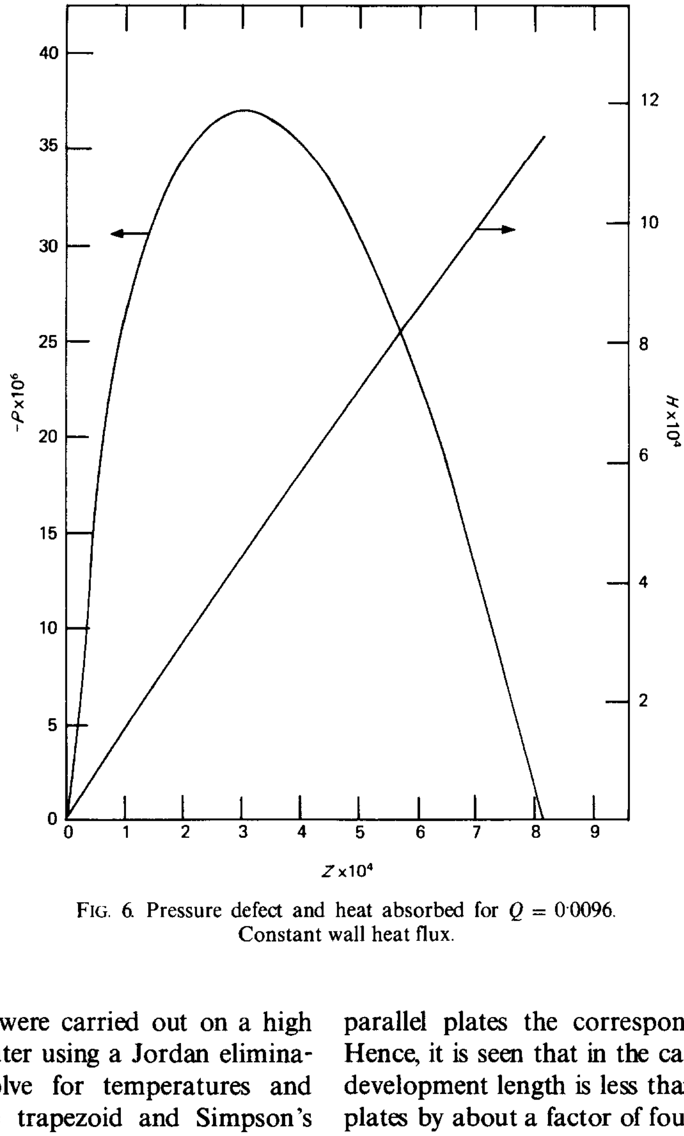

FIG. 6. Pressure defect and heat absorbed for $Q = 0{\cdot}0096$. Constant wall heat flux.

The calculations were carried out on a high speed digital computer using a Jordan elimination scheme to solve for temperatures and velocities. Both the trapezoid and Simpson's rules were used to perform the numerical integrations for F and Q with no significant differences in results.

Constant wall temperature

The variation of F and Q' with L is shown in Fig. 7. Also shown is the variation of dimensionless mixing-cup temperature (T'_m) with L. Here it is seen that for F equal to 90 per cent of the fully developed value ($\frac{1}{8}$) that L is approximately equal to 0·25 or Gr^* is equal to 4. For infinite parallel plates the corresponding L is 1 [4]. Hence, it is seen that in the case of the tube, the development length is less than that for parallel plates by about a factor of four when the radius of the tube is equal to half of the plate spacing.

The work was compared with the analytical and experimental work of Elenbaas [3]. Elenbaas presented his results as a plot of $Nu_{r,w}$ vs. Gr^*, Pr. The heat transfer coefficient was evaluated using the initial temperature difference. In terms of the dimensionless variables used here $Nu_r = (Q'Gr^*\,Pr)/2$.

Figure 8 shows the comparison of the experimental work of Elenbaas with the present work. It is of interest to note that for large values of

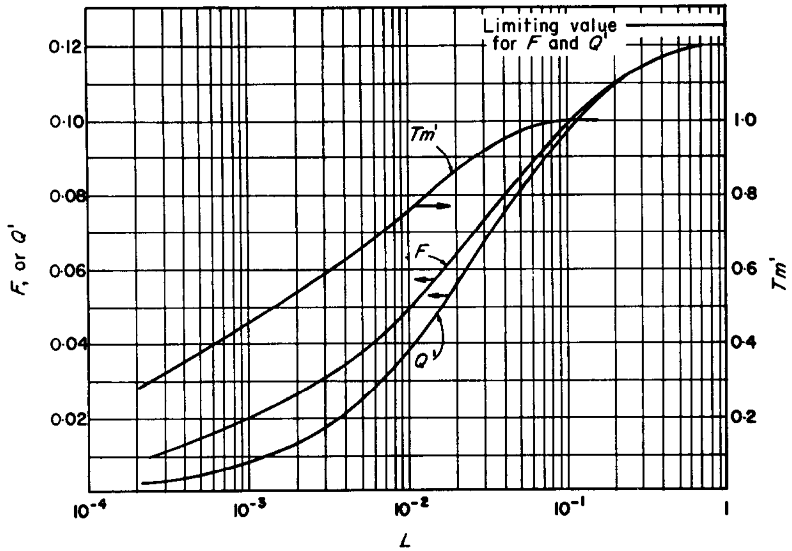

FIG. 7. Variation of dimensionless flow, heat absorbed and mixing cup temperature with dimensionless tube length. Constant wall temperature $Pr = 0{\cdot}7$.

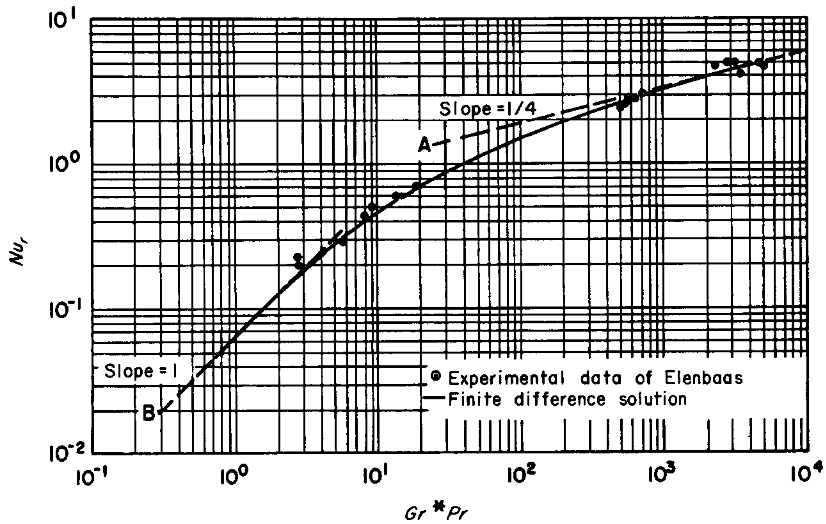

FIG. 8. Comparison of Nu_r vs. $Gr^*\ Pr$ with the data of Elenbaas [3]. Constant wall temperature.

Gr^*Pr, the slope of the curve approaches the $\frac{1}{4}$ power law of heated vertical plates. This would correspond to a large diameter tube where there is side development as opposed to center line development. The dotted line (A) on the graph can be represented by the equation

$$Nu_r = 0{\cdot}61\,(Gr^*\ Pr)^{\frac{1}{4}},$$

which compares rather well with the 0·60 $Gr^*\ Pr$ reported by Elenbaas.

For low values of $Gr^* Pr$ the equation describing Nu_r is that for fully developed flow, $Nu_r = \frac{1}{16} Gr^* Pr$, and is represented by line B on the graph.

Elenbaas obtained an approximate solution to the constant wall temperature problem by assuming that the radial velocity component v was zero. His analytical solution is

$$Nu_{r,w} = \tfrac{1}{16} Gr_w^* Pr \times \left\{1 - \exp\left[-\frac{1}{16}\left(\frac{5}{Gr^* Pr}\right)^{\frac{3}{4}}\right]\right\}.$$

The maximum deviation between his analytical results and the finite difference solution presented here was 13 per cent, and occurs at values of $Gr^* Pr$ of 10–100, where flow is neither approaching full development nor side development. It is in this range that neglect of v should be most serious.

Constant wall heat flux

The variation of F and Q' with L is shown in Fig. 9. As was expected, Q' varies linearly with L and can be represented by $Q' = 2L/Pr$. For this case there was no upper limiting value of F as was found for the condition of constant wall temperature since the driving force is sustained. Using $\partial U/\partial Z = 0$ as a criterion for hydrodynamically developed flow it was found that for F approaching $\frac{1}{8}$ the flow became developed before leaving the tube. Table 1 shows the comparison between the velocity profiles for $F = 0{\cdot}1353$ as obtained from the numerical solution and the analytical solution presented in the Theory.

Table 1. Comparison of analytical and numerical solution for velocity and temperatures—constant wall wall heat flux $F = 0{\cdot}1353$

R	U^*	$U\dagger$	$T - T_{cl}^*$	$T - T_{cl}\dagger$
0	0·25378	0·25578	0·00000	0·00000
0·1	0·25177	0·25309	0·00940	0·01224
0·2	0·24567	0·24567	0·03731	0·04029
0·3	0·23643	0·23618	0·07733	0·08544
0·4	0·22020	0·22095	0·14526	0·14672
0·5	0·20005	0 20061	0·22256	0·22253
0·6	0·17120	0·17460	0·32241	0·31072
0·7	0·14189	0·14226	0·41294	0 40841
0·8	0·10273	0·10287	0·51847	0·51205
0·9	0·05566	0·05570	0·62597	0·61731

† Analytical.
‡ Numerical.

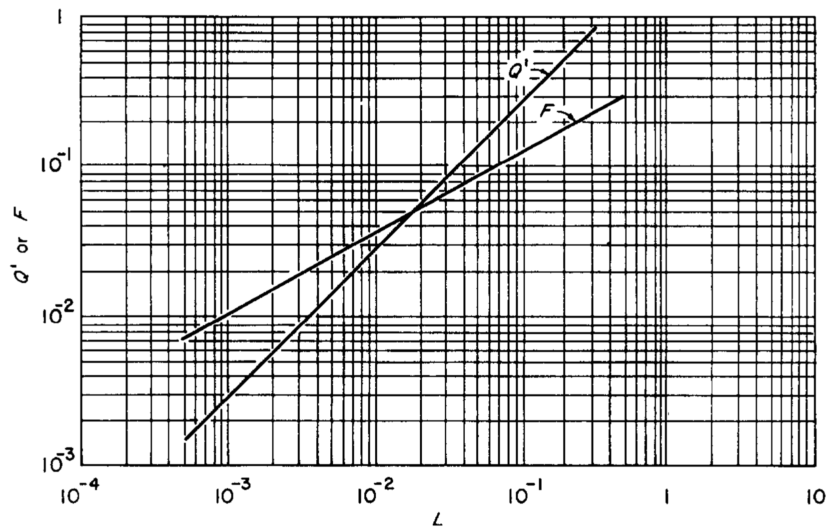

FIG. 9. Variation of dimensionless flow and heat absorbed with dimensionless tube length. Constant wall heat flux $Pr = 0{\cdot}7$.

The agreement is quite good. The comparison of $T - T_{cl}$ from the two methods is also presented; however, the agreement is not as good as for the velocity profiles.

Instead of plotting Nu vs. $Gr^* \; Pr$ as was done for constant wall temperature it was desired to compare the present work with that of Kays [8] on laminar forced convection. Toward this end the local Nusselt number based on diameter was plotted vs. $z/Re\,Pr\,D$, or in terms of the dimensionless variables used in this investigation, vs. $Z/4F\,Pr$. This comparison is shown in Fig. 4 for two extreme cases of F (0·01 and 0·2). In addition to the curve due to Kays which shows marked disagreement for small values of the abscissa there is presented the results of calculations in which the temperature dependence was omitted in the equation of motion (i.e. forced convection). This is also in disagreement with Kays' results and can be attributed to Kays' neglecting the transverse velocity component in his calculations. It is of interest to note that the two curves for natural convection have different asymptotic values for large values of the abscissa with the curve for the larger value of F being nearly identical with that obtained for forced convection. This is a result of the effect the heat transfer has on the velocity profiles. Small values of F (high heating rates and large radius) favor side development resulting in larger deviation from laminar forced convection profiles.

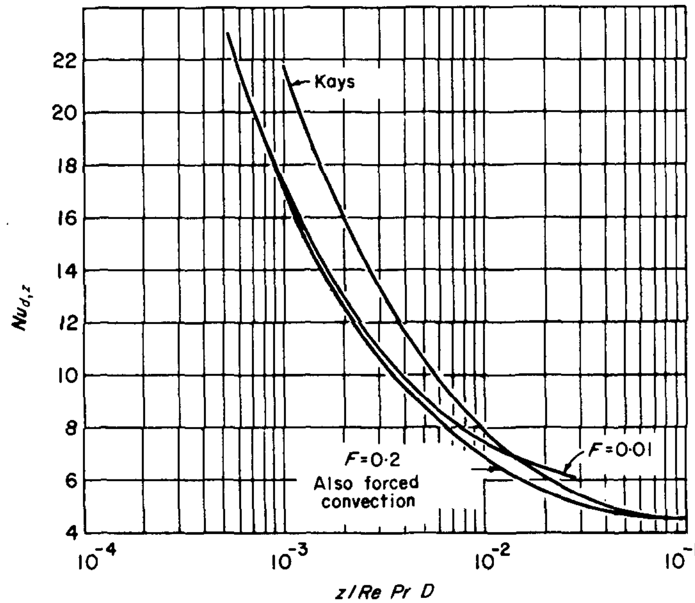

FIG. 10. Comparison of local Nusselt number with work of Kays [8]. Constant wall heat flux.

CONCLUSIONS

From the results of this investigation one can conclude that the development height for free convective flow in a heated open vertical tube is quite large and consequently causes the assumption of fully developed flow to be invalid for many situations involving constant wall temperature heat transfer. Fully developed flow is approached for very small values of the modified Grashof numbers.

The asymptotic behavior seen for constant wall temperature is not observed for constant wall heat flux because of the maintenance of the buoyancy driving force. The results indicate that for values of the dimensionless flow in excess of $\frac{1}{8}$ that fully developed flow exists—a situation that was only approached for constant wall temperature.

The inclusion of the transverse velocity component in the calculations for the developing velocity and temperature profiles has a significant influence on the Nusselt number correlations for both heating conditions.

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DÉVELOPPEMENT DE L'ÉCOULEMENT À CONVECTION LIBRE D'UN GAZ DANS UN TUBE CHAUFFÉ VERTICAL OUVERT

Résumé—Le système se compose d'un tube vertical ouvert aux deux extrémités dont on chauffe la paroi. Un gaz ambiant ($Pr = 0{\cdot}7$) pénètre par le fond du tube à des vitesses et températures uniformes et s'élève à travers le tube par convection libre.

On suppose l'écoulement à la fois stable et laminaire. Dans ce cas, les équations de la couche limite thermique incompressible sont résolues par une méthode aux différences finies, pour les conditions de température et de flux thermique constants à la paroi.

A partir des courbes de vitesse et température obtenues pour des différentes étapes du développement de l'écoulement on a trouvé des relations graphiques sans dimension entre la longueur du tube et deux quantités représentatives du débit volumique et du taux de dissipation thermique. Dans le cas d'une température constante à la paroi, on compare ces résultats à ceux de Elenbaas et on obtient un excellent accord. Quant aux résultats à flux pariétal thermique constant on les compare à ceux de Kays sur la convection forcée laminaire. Ses calculs dans lesquels la composante transversale de la vitesse est supposée négligeable donnent des nombres de Nusselt à l'entrée supérieurs de 20 à 30 pour cent aux résultats présents.

ENTWICKLUNG EINER FREIEN KONVEKTIONSSTRÖMUNG VON GAS IN EINEM SENKRECHTEN, OFFENEN, BEHEIZTEN ROHR

Zusammenfassung—Das System besteht aus einem senkrechten beidseitig offenen Rohr, dessen Wand beheizt ist. Ein umgebendes Gas ($Pr = 0{\cdot}7$) tritt am unteren Ende mit gleichmässiger Geschwindigkeit ein und strömt durch das Rohr auf Grund der freien Konvektion. Die Strömung wird als stationär und laminar angenommen. Die hier gültigen Gleichungen der thermischen Grenzschicht wurden mit Hilfe der Differenzenmethode bei konstanter Wandtemperatur und konstantem Wärmestrom gelöst.

Aus den bei verschiedenen Strömungen erhaltenen Geschwindigkeits- und Temperaturprofilen wurde eine grafische Korrelation entwickelt zwischen dimensionsloser Rohrlänge und zwei dimensionslosen Werten, die den Massenstrom und die Wärmedissipation charakterisieren. Für konstante Wandtemperatur wurden die Ergebnisse mit jenen von Elenbaas (3) verglichen, wobei sich sehr gute Übereinstimmung zeigte. Die Ergebnisse für konstanten Wärmefluss wurden mit jenen von Kays (8) für laminare Zwangskonvektion verglichen. Seine Berechnungen, in denen die umgekehrte Geschwindigkeitskomponente vernachlässigt wurde, ergab Nusseltzahlen nahe dem Einlauf, die 20 bis 30 Prozent höher liegen als die hier erhaltenen Werte.

HHH

РАЗВИТИЕ СВОБОДНОЙ КОНВЕКЦИИ ПРИ ТЕЧЕНИИ ПОТОКА ГАЗА В ОТКРЫТОЙ ВЕРТИКАЛЬНОЙ ТРУБЕ

Аннотация—Система состоит из вертикальной трубы, открытой с обеих концов. Окружающий газ (Pr = 0.7) поступает в нижнюю часть трубы с постоянной скоростью и течет по ней вверх вследствие естественной конвекции. Считается, что поток является и устойчивым и ламинарным. Уравнения для несжимаемого теплового пограничного слоя для этого случая были решены методом конечных разностей при условиях постоянной температуры стенки и постоянного теплового потока.

На основании профилей скорости температуры, полученных для различных стадий развития потока, установлено графическое соотношение между безразмерной длиной трубы и двумя безразмерными величинами, представляющими объемную скорость потока и скорость рассеяния тепла. Для случая постоянной температуры эти результаты сравнивались с результатами Эленбаса (3), причем данные согласуются отлично. Результаты исследования постоянного теплового потока стенки сравнивались с результатами Кейса (8) по ламинарной вынужденной конвекции. Его расчеты, при которых поперечной составляющей пренебрегали, дали числа Нуссельта вблизи входа, которые были на 20–30% выше, чем представленные результаты.